Lattices of Annihilators in Commutative Algebras

Małgorzata Jastrzębska

Siedlce University of Natural Sciences and Humanities

(joint work with Jan Krempa
University of Warsaw)

Noncommutative rings and their applications IV,
Lens, June 8-11, 2015

In this notes $\mathbb{K}$ will be a field. All algebras are associative $\mathbb{K}$-algebras, with $1 \neq 0$. If A is an algebra, then A^{op} denotes the algebra with the same linear structure over $\mathbb{K}$ but with the opposite multiplication.

All lattices have the smallest element ω and the largest element $\Omega \neq \omega$. If L is a lattice then by L^{op} we denote the lattice with the reverse order.

Agreements

For every algebra A the set $\mathcal{I}_l(A)$ of all left ideals and the set $\mathcal{I}_r(A)$ of all right ideals, ordered by inclusion are complete, modular lattices with operations:

$$I \vee J = I + J \quad \text{and} \quad I \wedge J = I \cap J. \tag{1}$$

In these lattices $\omega = 0$ and $\Omega = A$.

$$\mathcal{I}_l(A^{op}) = \mathcal{I}_r(A) \quad \text{and} \quad \mathcal{I}_r(A^{op}) = \mathcal{I}_l(A). \tag{2}$$

Agreements

If $X \subseteq A$ is a subset, then let $\mathrm{L}_A(X) = \mathrm{L}(X)$ be the left annihilator of X in A and let $\mathrm{R}_A(X) = \mathrm{R}(X)$ be the right annihilator of X in A :

$$\mathrm{L}(X) = \{a \in A : aX = 0\}, \tag{3}$$

$$\mathrm{R}(X) = \{a \in A : Xa = 0\}. \tag{4}$$

Then $X \subseteq \mathrm{L}(\mathrm{R}(X)), \quad \text{and} \quad X \subseteq \mathrm{R}(\mathrm{L}(X)).$

Let $\mathcal{A}_l(A)$ be the set of all left annihilators in A and $\mathcal{A}_r(A)$ be the set of all right annihilators in A.

The set $\mathcal{A}_l(A) \subseteq \mathcal{I}_l(A)$ is a complete lattice with operations:

$I \vee J = \mathrm{L}(\mathrm{R}(I) \cap \mathrm{R}(J)) \quad \text{and} \quad I \wedge J = I \cap J$,
for $I, J \in \mathcal{A}_l(A)$.

Similarly, $\mathcal{A}_r(A) \subseteq \mathcal{I}_r(A)$ is a complete lattice with operations:

$I \vee J = \mathrm{R}(\mathrm{L}(I) \cap \mathrm{L}(J)) \quad \text{and} \quad I \wedge J = I \cap J$,
for $I, J \in \mathcal{I}_r(A)$

$$\mathcal{A}_l(A^{op}) = \mathcal{A}_r(A) \quad \text{and} \quad \mathcal{A}_r(A^{op}) = \mathcal{A}_l(A).$$

There is a Galois correspondence between $\mathcal{A}_l(A)$ and $\mathcal{A}_r(A)$:
$\mathcal{A}_l(A) \overset{\mathrm{R}}{\simeq} (\mathcal{A}_r(A))^{op}, \quad (\mathcal{A}_r(A))^{op} \overset{\mathrm{L}}{\simeq} \mathcal{A}_l(A)$.

Let S be a semigroup and let I be an ideal in S. Then the Rees factor semigroup S/I may be identified with the set $(S \setminus I) \cup 0$ subject to the multiplication $\circ$ defined by the formula

$$s \circ t = \begin{cases} st & \text{if } st \notin I \\ 0 & \text{if } st \in I \end{cases}$$

Let S be a semigroup. The semigroup algebra $\mathbb{K}[S]$ is a $\mathbb{K}$-space with the basis S and the multiplication induced by the multiplication in S.

Let S be a semigroup with zero 0. By the contracted semigroup algebra of S over $\mathbb{K}$, denoted by $\mathbb{K}_0[S]$, we mean the factor algebra $\mathbb{K}[S]/\mathbb{K}0$.

Basic example I

Let P be a nonempty poset and let $\mathrm{M}(P)$ be the free monoid with the set P of free generators. Consider in $\mathrm{M}(P)$ an ideal I generated by all products xyz where $x, y, z \in P$ and all products xy where $x, y \in P$ and $x \leq y$. Put $\overline{P} = \mathrm{M}(P)/I$, the Rees factor monoid.
Now let $\mathbb{K}(P) = \mathbb{K}_0[\overline{P}]$ be the contracted monoid algebra. Thus $P \subset \mathbb{K}(P)$ and $\mathbb{K}(P)$ has the natural gradation given by:

$$\mathbb{K}(P) = \mathbb{K} \oplus V \oplus V^2, \tag{5}$$

where the natural base of V can be identified with P and the natural base of V^2 can be identified with
$\{xy : x, y \in P,\ x \not\leq y\}$.

If P is a finite poset, then the algebra $\mathbb{K}(\mathrm{P})$ is finite dimensional.

Theorem

Let P be any poset and let $\phi : P \longrightarrow \mathcal{A}_I(\mathbb{K}(P))$ be given by $\phi(x) = \mathrm{L}_{\mathbb{K}(P)}(x)$ for $x \in P$. Then ϕ is an embedding and preserves all existing meets and joins.
If L is a complete lattice, then ϕ is lattice isomorphism of L with the interval $[\phi(\omega), \phi(\Omega)] \subseteq \mathcal{A}_I(\mathbb{K}(L))$.

Corollary

Let L be a finite lattice such that it has exactly one atom and exactly one coatom. Then there exists an algebra A such that $L \simeq \mathcal{A}_I(A)$.

Basic Example in Commutative case

Let P be a nonempty poset and let P' be the set such that $|P| = |P'|$. Let $f : P \longrightarrow P'$ be a bijection given by $f(x) = x' \in P'$. Put $P^* = P \cup P'$. Let $S(P)$ be the free, commutative monoid with the set P^* of free generators. Consider in $S(P)$ an ideal I generated by all products xyz where $x, y, z \in P^*$ and by all elements of the set $\{x'y \mid x, y \in P \text{ where } x \leq y\}$. Put $\overline{S} = S(P)/I$, the Rees factor monoid.
Now let $\mathbb{K}(\!(P)\!) = \mathbb{K}_0[\overline{S}]$ be the contracted monoid algebra.

Basic Example in Commutative case

Thus $P \subset \mathbb{K}((P))$ and $\mathbb{K}((P))$ has the natural gradation given by:

$$\mathbb{K}((P)) = \mathbb{K} \oplus W \oplus W^2, \tag{6}$$

where the natural base of W can be identified with P^* and the natural base of W^2 can be identified with $(P^*)^2 = \{xy | x, y \in P\} \cup \{x'y' | x', y' \in P'\} \cup \{x'y \mid x, y \in P, x \not\leq y\}$.

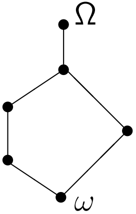
Our algebra $\mathbb{K}((P))$ is a local, commutative algebra with the radical $J = W \oplus W^2$ and with the residue field $\mathbb{K}((P))/J = \mathbb{K}$.

If P is a finite poset, then algebra $\mathbb{K}((P))$ is finite dimensional.

Let A be a commutative algebra. Then $\mathrm{R}(X) = \mathrm{L}(X)$ for any subset $X \subseteq A$. Thus $\mathcal{A}_l(A) = \mathcal{A}_r(A)$. We put $\mathrm{R}(X) = \mathrm{L}(X) = \mathrm{a}(X)$ and $\mathcal{A}_l(A) = \mathcal{A}_r(A) = \mathcal{A}(A)$.
Using the above notations we have

Theorem

Let P be nonempty poset and let $\psi : P \longrightarrow \mathcal{A}(\mathbb{K}(\!(P)\!))$ be given by $\psi(x) = \mathrm{L}_{\mathbb{K}(\!(P)\!)}(x)$ for $x \in P$. Then ψ is an embedding and preserves all existing meets and joins.
If L is a complete lattice then L is isomorphic to sublattice of the lattice $\mathcal{A}(\mathbb{K}(\!(P)\!))$.



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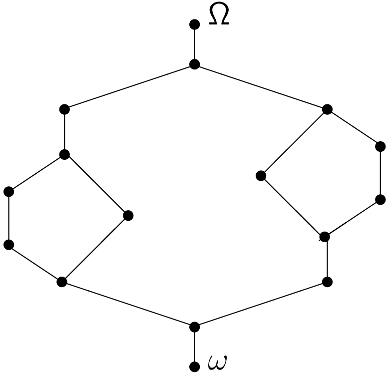

$\mathcal{A}_I(\mathbb{K}((\mathbb{L})))$

If a lattice identity holds in each finite lattice, then this identity holds in every lattice.

Corollary

There is no nontrivial lattice identity satisfied in lattices of annihilators of all commutative, finite dimensional algebras.

Full text of this work will appear in Demonstratio Mathematica, probably in vol. 48(2015).

Thank you for your attention!